

# Limitations of practical multi-photon decoherence-free states

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It is shown in this paper that decoherence-free subspace (DFS) of practical multi-photon polarization can not avoid the exponential decoherence even in the same extra-environment if the photons are frequency-anticorrelated. The reason lies in that the condition of collective decoherence is not satisfied in this case. As an example, the evolution of biphoton's decoherence-free state is given. Possible solution for feasible multi-photon's DFS state is also given.

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## I. INTRODUCTION

Quantum information processing provides secure communication [1,2] and powerful quantum communication such as quantum teleportation [3]. The advantages of these tasks originate from the superposition of quantum states [4]. However, superposition is very fragile and easily destroyed by the decoherence process due to unwanted coupling with the environment [5]. Several strategies have been devised to cope with decoherence, each of them is appropriate for a specific type of coupling with the environment [6]. The first, Quantum error correcting code [7], relies on trying to detect errors using ancillary quantum bits (qubits) and actively manipulating the interactions to correct these errors. The second strategy employs dynamical decoupling [8,9], in which rapid switching is used to average out the effects of a relatively slowly decohering environment. The final approach attempts to embed the logical qubits into a part of the overall Hilbert space that is inherently immune to noise, a decoherence-free subspace (DFS) [10,11]. Beside the general solutions mentioned above, another strategy particularly for long-distance quantum communication has been also presented [12], i.e. quantum repeater which combines entanglement purification and entanglement swapping to build reliable communication channel.

Photons are obvious candidates for mediators of quantum information particularly in quantum communication since they are fast, achieved easily, and interact weakly with the environment. The degree of freedom used to encode the information can be the polarization of the photon, its phase, or some combination of both. In decoherence process of photons, polarization effects are important source of problems, e.g. polarization-mode dispersion in optical fibre [13,14].

Quantum error correcting code is not suitable to overcome the photon's decoherence, since it needs at least five qubits to encode a logic qubit and correct only single-bit error. Though five-photon entangled state has been realized [15], achieving a five-photon state, encoding and decoding it definitely is a hard task in recent future. The dynamical decoupling strategy is also not so easy to be realized as the similar reasons. Entanglement purification in quantum repeater also needs many steps of quantum operations. Linear optical quantum repeater, which has been demonstrated in experiment [16], however needs all entangled photon pairs are produced from the common sources, and this is not practical for long-distance communication. It seems that DFS is a good method to conquer the decoherence since it only needs two qubits to encode a state and four qubits to encode a logic qubit and it does not need so many quantum operations between different qubits. In fact, some schemes based on DFS have been demonstrated by photons [17,18] or other systems [19] and much more promising quantum information protocols based on photon's DFS states have been proposed, such as robust quantum key distribution protocol [20], communication without a shared reference frame [21,22] etc.

Practical entangled photons which are mainly produced by spontaneous parametric down-conversion (SPDC) are frequency-anticorrelated [23], so they do not experience collective decoherence in communication medium (e.g. optical

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fibre) which is the necessary condition of DFS. Hence, it is shown in this paper that though frequency-anticorrelated multi-photon can be encoded into a DFS and experience the same extra-environment, they can not avoid the exponential decoherence. The reason lies in that they have different intra-environment.

This paper is organized as follows: in Sec. II, we describe the single photons decoherence in a practical way by the method of Master Equation [24]; in Sec. III, the decoherence of a frequency-anticorrelated biphoton's DFS state is shown; in Sec. IV, we discuss the general cases of decoherence of multi-photon's DFS states and in Sec. V we summarize the conclusion.

## II. SINGLE PHOTON'S DECOHERENCE

A single-photon's initial pure state can be represented by a pure product state of polarization and frequency [17],

$$|\Psi\rangle = (a_1 |H\rangle + a_2 |V\rangle) \otimes \int d\omega A(\omega) |\omega\rangle \quad (1)$$

with basis  $|H\rangle$  (horizontal polarization) and  $|V\rangle$  (vertical polarization) denoted by

$$|H\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

respectively, and  $A(\omega)$  is the complex amplitude corresponding to  $\omega$ , normalized so that

$$\int d\omega |A(\omega)|^2 = 1.$$

Here we assume that the frequency spectrum has a Gaussian form

$$A(\omega) = \pi^{-\frac{1}{4}} \sqrt{\frac{1}{\delta}} \exp \left[ -\frac{(\omega - \omega_0)^2}{2\delta^2} \right].$$

The stochastic coupling of photon's polarization with frequency via anisotropic medium (e.g. non-ideal single-mode fibre) can be viewed as many local birefringence processes of random direction [13]. The Hamiltonian that describes this coupling can be written as

$$H_I = g(\omega) (\omega - \omega_0) \left[ \sigma_x \sum_{\mathbf{k}_1} (a_{\mathbf{k}_1} + a_{\mathbf{k}_1}^\dagger) + \sigma_y \sum_{\mathbf{k}_2} (a_{\mathbf{k}_2} + a_{\mathbf{k}_2}^\dagger) + \sigma_z \sum_{\mathbf{k}_3} (a_{\mathbf{k}_3} + a_{\mathbf{k}_3}^\dagger) \right]. \quad (2)$$

Here  $\sigma_{x,y,z}$  are Pauli operators,  $g(\omega)$  is assume to be a constant  $g$  and the three bathes denoted by  $\mathbf{k}_1, \mathbf{k}_2$  and  $\mathbf{k}_3$  respectively have the same statistic properties. For a Markovian reservoir [24], we can obtain the master equation of the single photon's polarization as follows.

$$\frac{\partial \rho}{\partial t} = \frac{\gamma}{2} (\sigma_x \rho \sigma_x + \sigma_y \rho \sigma_y + \sigma_z \rho \sigma_z - 3\rho). \quad (3)$$

Here  $\gamma$  is a constant which is dependent on the medium. The solution of this equation is

$$\rho(t) = \rho(0) e^{-2\gamma t} + \frac{1}{2} I (1 - e^{-2\gamma t}), \quad (4)$$

where  $I$  is the  $2 \times 2$  identity matrix. The single photon's polarization state will be decohered to the complete mixed state exponentially.

## III. DECOHERENCE OF TWO FREQUENCY-ANTICORRELATED PHOTONS' DFS STATE

The two frequency-anticorrelated photons' DFS state can be described as

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|HV\rangle - |VH\rangle)_{12} \otimes \int d\omega B(\omega) |\omega_0 + \omega\rangle_1 |\omega_0 - \omega\rangle_2, \quad (5)$$

where  $\omega_0$  is equal to half of centre frequency of the pump and we assume the down-conversion photons have a Gaussian spectrum (in fact it depends on the filter)

$$B(\omega) = \pi^{-\frac{1}{4}} \sqrt{\frac{1}{\delta}} \exp\left[-\frac{\omega^2}{2\delta^2}\right].$$

Here we neglect the width of the pump's spectrum, because it has no effect on the interaction Hamiltonian of polarization and frequency. Particularly, in the case of continuous pump such as Ar ion laser, the pump's spectrum width is far narrower than the down-conversion photon's width. The Hamiltonian that describes the coupling of two photons with the same medium is

$$H_I' = g \sum_i \omega (\sigma_{i,1} - \sigma_{i,2}) \sum_{\mathbf{k}_i} (a_{\mathbf{k}_i} + a_{\mathbf{k}_i}^\dagger), \quad (6)$$

where  $i = x, y, z$  and  $\sigma_{i,1}(\sigma_{i,2})$  are Pauli operators of photon 1(2). For a Markovian reservoir, we can obtain the master equation of the two photons' polarization as follows

$$\begin{aligned} \frac{\partial \rho}{\partial t} = \gamma \sum_i [ & (\sigma_{i,1} - \sigma_{i,2}) \rho (\sigma_{i,1} - \sigma_{i,2}) \\ & - (\sigma_{i,1} - \sigma_{i,2}) (\sigma_{i,1} - \sigma_{i,2}) \rho - \rho (\sigma_{i,1} - \sigma_{i,2}) (\sigma_{i,1} - \sigma_{i,2}) ]. \end{aligned} \quad (7)$$

For an initial state of Eq. (5), the solution of the master equation is

$$\rho(t) = \frac{1}{4} (1 - e^{-8\gamma t}) I_4 + \rho(0) e^{-8\gamma t}, \quad (8)$$

where  $I_4$  is the identity operator for a 4-dimension system and  $\rho(0)$  is the polarization state of Eq. (5). The stationary state is  $\rho = \frac{1}{4} I_4$ , any deviation from it will vanish exponentially. Details of the solution are shown in the Appendix. It can be seen that the exponential decoherence is inevitable and there is none DFS state under the Hamiltonian shown in Eq. (6).

#### IV. DECOHERENCE OF GENERAL DFS STATES

The evolution of an  $N$ -photon's polarization state in the same medium can be described by the following Hamiltonian

$$H_I'' = g \sum_{i=1}^3 \sum_{j=1}^N \Delta_j \sigma_{i,j} \sum_{\mathbf{k}_i} (a_{\mathbf{k}_i} + a_{\mathbf{k}_i}^\dagger), \quad (9)$$

where  $\Delta_j = \omega_j - \omega_0$  is the deviation from its centre frequency of the  $j$ th photon and  $\sigma_{i,j}$  are Pauli operators of the  $j$ th photon. The master equation under Markovian approximation is

$$\frac{\partial \rho}{\partial t} = \gamma' \sum_{i=1}^3 \sum_{j,k=1}^N \overline{\Delta_j \Delta_k} (\sigma_{i,j} \rho \sigma_{i,k} - \sigma_{i,j} \sigma_{i,k} \rho - \rho \sigma_{i,j} \sigma_{i,k}). \quad (10)$$

It can be verified that the DFS state keeps invariant only when  $\Delta_j = 0$  holds for any  $j$ . For example, in a four-qubit DFS the logic qubit is denoted as

$$\begin{aligned} |0_L\rangle &= \frac{1}{2} (|HV\rangle - |VH\rangle)_{12} (|HV\rangle - |VH\rangle)_{34}, \\ |1_L\rangle &= \frac{1}{\sqrt{3}} (|HHVV\rangle + |VVHH\rangle)_{1234} \\ &\quad - \frac{1}{2\sqrt{3}} (|HV\rangle + |VH\rangle)_{12} (|HV\rangle + |VH\rangle)_{34}. \end{aligned}$$

There are  $H_I'' |0_L\rangle \neq 0$  and  $H_I'' |1_L\rangle \neq 0$  unless  $\Delta_j = 0$  for any  $j$ .

## V. DISCUSSION AND SUMMARY

The most important condition required by DFS is the collective decoherence, i.e. all qubits must experience the same environment and have the same coupling with the environment. In the coupling of photons with dispersive medium, the environment includes medium and the intra environment, i.e. the frequency freedom of photons. However, frequencies of two photons from SPDC are anticorrelated which does not satisfy the condition of collective decoherence.

There are two methods to solve this problem. The first is to let the coupling coefficients of two photons are inverse everywhere which has been used to demonstrate DFS in experiment [17], but it is not practical in the stochastic medium for long-distance quantum communication. The second method is to use frequency-correlated photon pairs. Since the conservation of energy, producing frequency-correlated multi-photon states seems to be a hard problem. However, only a part of the photons are frequency-correlated also can let the conservation of energy hold. For example, we can produce the four-photon state in SPDC process, but only use two frequency-correlated photons of all photons. Though the efficiency decreases, we can also fulfil many interesting tasks in long-distance quantum communication [20–22]. Another method for producing frequency-correlated photons has been proposed [25]. It is essentially to produce a narrow-spectrum but not true frequency-correlated photon pair, and it may be useful in the situation which has not too high requirement in long-distance quantum communication.

Note that we have neglected the spectrum width of the pump in the SPDC process. But there is not any positive result if it is accounted, although it has no effect on the coupling, it will cause dispersion independent on the polarization.

The frequency distribution also exists in the coupling of other systems such as atom in cavity, ion trap and solid qubit systems with the environment. It is an open question whether it will affect the efficiency of the DFS in these systems.

In conclusion, considering the photon's other freedom is involved in the coupling with the environment, the photons may not decohered collectively though all of them pass through the same communication medium i.e. all paths of decoherence do not cancel each other. Therefore they can not avoid the exponential decoherence and it is a drawback for long-distance quantum communication. However, frequency-correlated photon pairs may be useful to solve this problem.

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## Appendix: Calculation of the decoherence of two-photon's DFS state

The Eq. (7) can be written in this form

$$\begin{aligned}
\frac{1}{\gamma} \frac{\partial \rho_{11}}{\partial t} &= -2\rho_{11} + \rho_{22} + \rho_{33} - \rho_{23} - \rho_{32}, \\
\frac{1}{\gamma} \frac{\partial \rho_{12}}{\partial t} &= -3\rho_{12} + \rho_{13} - \rho_{24} + \rho_{34}, \\
\frac{1}{\gamma} \frac{\partial \rho_{13}}{\partial t} &= -3\rho_{13} + \rho_{12} + \rho_{24} - \rho_{34}, \\
\frac{1}{\gamma} \frac{\partial \rho_{14}}{\partial t} &= -2\rho_{14}, \\
\frac{1}{\gamma} \frac{\partial \rho_{22}}{\partial t} &= -2\rho_{22} + \rho_{11} + \rho_{44} + \rho_{23} + \rho_{32}, \\
\frac{1}{\gamma} \frac{\partial \rho_{23}}{\partial t} &= -\rho_{11} + \rho_{22} + \rho_{33} - 6\rho_{23} - \rho_{44}, \\
\frac{1}{\gamma} \frac{\partial \rho_{24}}{\partial t} &= -\rho_{12} + \rho_{13} - 3\rho_{24} + \rho_{34}, \\
\frac{1}{\gamma} \frac{\partial \rho_{33}}{\partial t} &= -2\rho_{33} + \rho_{11} + \rho_{44} + \rho_{23} + \rho_{32},
\end{aligned} \tag{A-1}$$

$$\begin{aligned}\frac{1}{\gamma} \frac{\partial \rho_{34}}{\partial t} &= \rho_{12} - \rho_{13} + \rho_{24} - 3\rho_{34}, \\ \frac{1}{\gamma} \frac{\partial \rho_{44}}{\partial t} &= -2\rho_{44} + \rho_{22} + \rho_{33} - \rho_{23} - \rho_{32}.\end{aligned}$$

Note here

$$\rho_{ij} = \rho_{ji}^* \quad (\text{A-2})$$

for any  $i, j$  and

$$\sum_{i=1}^4 \rho_{ii} = 1. \quad (\text{A-3})$$

To solve these equations, we can define the following variables

$$\begin{aligned}x_1 &= \rho_{11} - \rho_{44}, \\ x_2 &= \rho_{22} - \rho_{33}, \\ x_3 &= \rho_{11} - \rho_{22} - \rho_{33} + \rho_{44} + 2\rho_{23} + 2\rho_{32}, \\ x_4 &= \rho_{11} - \rho_{22} - \rho_{33} + \rho_{44} - \rho_{23} - \rho_{32}, \\ x_5 &= \rho_{23} - \rho_{32}, \\ x_6 &= \rho_{12} + \rho_{13}, \\ x_7 &= \rho_{24} + \rho_{34}, \\ x_8 &= \rho_{12} - \rho_{13} + \rho_{24} - \rho_{34}, \\ x_9 &= \rho_{12} - \rho_{13} - \rho_{24} + \rho_{34}, \\ x_{10} &= \rho_{14}.\end{aligned} \quad (\text{A-4})$$

There are

$$\begin{aligned}\dot{x}_1 &= -2\gamma x_1, \quad x_1 = x_1(0) e^{-2\gamma t}; \\ \dot{x}_2 &= -2\gamma x_2, \quad x_2 = x_2(0) e^{-2\gamma t}; \\ \dot{x}_3 &= -8\gamma x_3, \quad x_3 = x_3(0) e^{-8\gamma t}; \\ \dot{x}_4 &= -2\gamma x_4, \quad x_4 = x_4(0) e^{-2\gamma t}; \\ \dot{x}_5 &= -6\gamma x_5, \quad x_5 = x_5(0) e^{-6\gamma t}; \\ \dot{x}_6 &= -2\gamma x_6, \quad x_6 = x_6(0) e^{-2\gamma t}; \\ \dot{x}_7 &= -2\gamma x_7, \quad x_7 = x_7(0) e^{-2\gamma t}; \\ \dot{x}_8 &= -6\gamma x_8, \quad x_8 = x_8(0) e^{-6\gamma t}; \\ \dot{x}_9 &= -2\gamma x_9, \quad x_9 = x_9(0) e^{-2\gamma t}; \\ \dot{x}_{10} &= -2\gamma x_{10}, \quad x_{10} = x_{10}(0) e^{-2\gamma t}.\end{aligned} \quad (\text{A-5})$$

Combining equations from (A-2) to (A-5) we can obtain  $\rho_{ij}(t)$  for any  $i, j$ . The stationary state  $\rho = \frac{1}{4}I_4$  can be obtained if we let  $x_{1,\dots,10} = 0$ , any deviation from it will vanish exponentially.

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